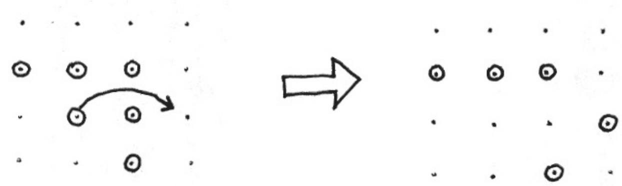
The Peg Solitaire (a.k.a. Hi-Q) Problem

► FIRST, the problem:

You have an infinite grid of holes in rows and columns. In the middle of that grid, you have n^2 of those holes filled by pegs in an $n \times n$ square; the rest of the holes are empty.

To make a move in Peg Solitaire, you pick up a peg and jump it over an adjacent peg (horizontally or vertically, not diagonally) into an empty hole. Then you remove the peg that was jumped over. For example:



You win if you can make a series of moves that leaves just one peg on the board.

For what values of n is it possible to win?

$$n \in \{??\}$$

This doodle was done by Ping on 2005-09-27.
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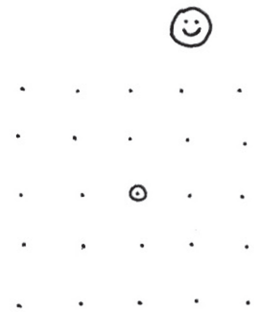
read on... →

► LET'S TRY SOME CASES.

How about $n = 1$?

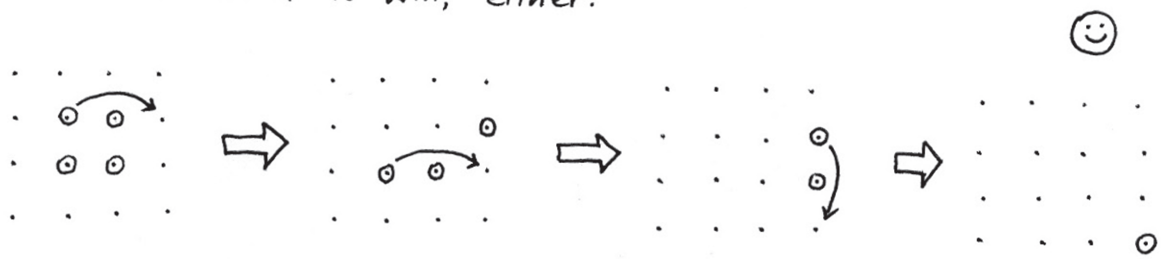
Well, that's easy. We've already won.



$n = 1$ ✓

How about $n = 2$?

This isn't hard to win, either.



$n = 2$ ✓

All right, then. How about $n = 3$?

If you try starting with a 3×3 square, you'll find it hard to end up with just one peg. In fact, it's impossible.

I owe you an apology for saying that 3×3 is winnable. I was wrong.

$n = 3$ ✗

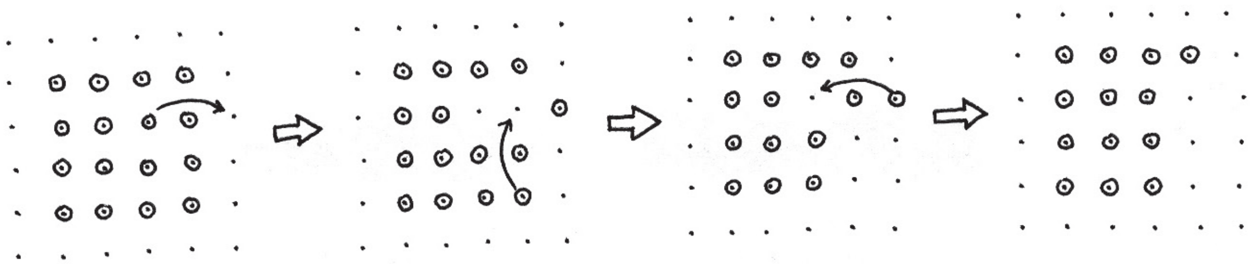
But if you try $n = 4$...

It turns out that $n = 4$ really <u>is</u> winnable. I'm not lying this time. I really mean it. Give it a shot yourself, before reading on to see how.

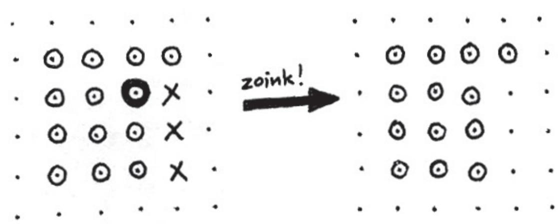
$n = 4$ ✓

HOW TO WIN $n = 4$

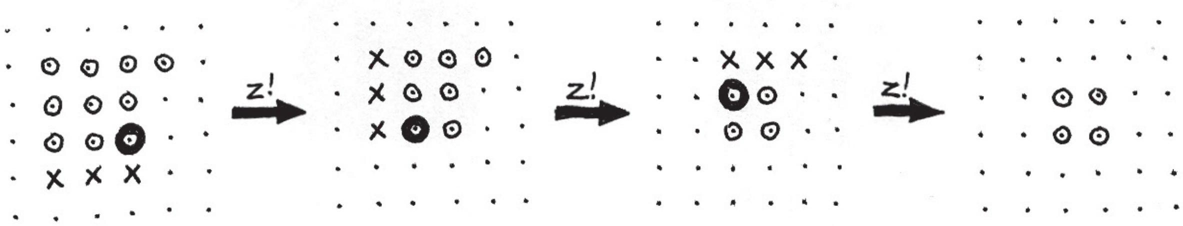
Here's one way to do it. Start with these three moves:



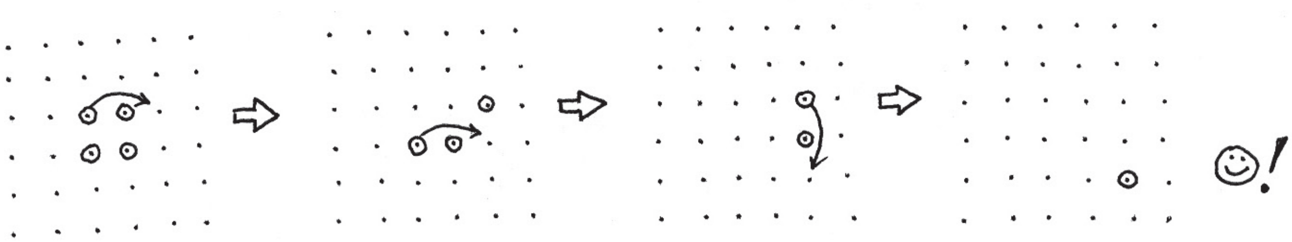
Let's call this three-move sequence a ZOINK. When we zoink, we remove three pegs in a row (marked ×) and the only other peg we need to use is the one marked ●.



We can do another zoink, rotated 90°. And another. And another.



And now we're back to a 2×2 square, which is easy to win.



Voilà! We're done. (It's possible to win when $n=5$, too, using artfully chosen zoinks. See if you can figure it out.)

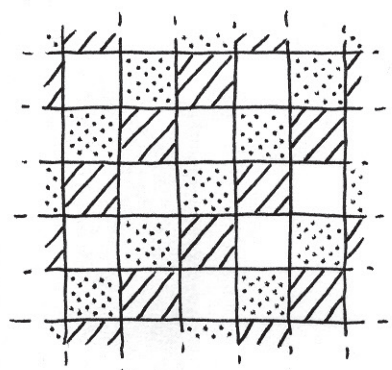
► OK, BUT WHY?

So far we know it works for $n \in \{1, 2, 4, 5\}$.
Can we prove it's impossible for $n=3$? How about $n=6$?

The trick is to colour the board...

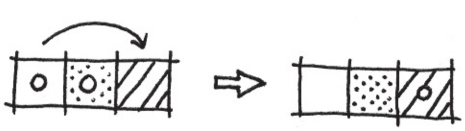
The number 3 has a special significance to this problem: we have a way to remove pegs in threes, it seems to be winnable for small values of n except 3, and each move involves three locations (the spot jumped from, the spot jumped to, and the spot jumped over). So:

... in three colours.

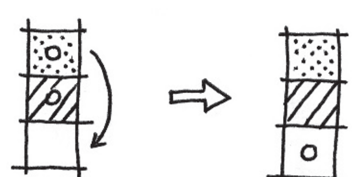


I'll draw squares instead of dots now, so we can shade them in. The pegs, still drawn as little circles, go inside the squares.

In every row and column, the colours go in a cycle: blank, dotty, stripey, blank, dotty, stripey, and so on. Suppose we keep track of how many pegs are in squares of each colour. Look at what happens when we make a move:



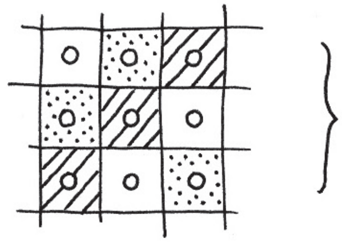
−1 blank
−1 dotty
+1 stripey



+1 blank
−1 dotty
−1 stripey

It doesn't matter what move we make.
Any legal move is going to remove a peg from two of the colours and add a peg to the third colour: always ± 1 for each colour.

If we start with 3×3 pegs, there will always be 3 pegs in each colour. 3 is an odd number.



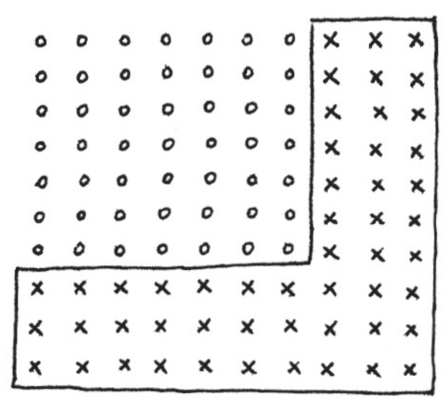
Every time we make a move, we'll add ± 1 to the number of pegs in each colour. Adding ± 1 always makes an odd number even, or makes an even number become odd.

After the first move, there will be even numbers for blank, dotty, and stripey squares. After the second move, they will all be odd numbers again, and so on.

But to win, we have to end up with just one peg in one colour and no pegs in the other colours. That can't ever happen if the three numbers are always all even or all odd. So that proves there really is no way to win when $n=3$.

In fact, whenever n is a multiple of 3, you'll start out with the same number of pegs in each colour. It's always impossible to win when n is a multiple of 3.

What about when n isn't a multiple of 3? Well, we know it's possible to win when n is 4 or 5. To show that you can win for any bigger non-multiples-of-3, all you need to do is come up with a sequence of zoinks that will get rid of the last three rows and last three columns of any square of pegs with $n > 6$. I'll leave that to you. Have fun!



fin.